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Mourre's theory for some unbounded Jacobi matrices

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Abstract

We show a Mourre estimate for a class of unbounded Jacobi matrices. In particular, we deduce the absolute continuity of the spectrum of such matrices. We further conclude some propagation theorems for them.

Keywords: Mourre estimate; Jacobi matrix; Absolutely continuous spectrum

1. Introduction

1.1. Overview

Let $\mathcal{H} := l^2(\mathbb{Z})$ be the complex Hilbert space of square summable sequences endowed with the scalar product

$$\langle \phi, \psi \rangle = \sum_{n \in \mathbb{Z}} \overline{\phi}_n \psi_n.$$

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Consider the second-order difference operator H acting in $\mathcal{H}$ by

$$(H\psi)_n = a_{n-1}\psi_{n-1} + b_n\psi_n + a_n\psi_{n+1}, \tag{1.1}$$

where $b_n \in \mathbb{R}$ and $a_n > 0$ for all $n \in \mathbb{Z}$. It is usually called a bilateral Jacobi matrix. Clearly, H is a densely defined symmetric operator on $l_0^2(\mathbb{Z})$. The latter denotes the subspace of sequences with only finitely many nonzero coordinates. It is known that (cf. [2]), in general, H is not essentially self-adjoint on $l_0^2(\mathbb{Z})$ if the weights a_n grow fastly to infinity. In such case all the self-adjoint extensions have a purely discrete spectrum. In the sequel we assume that H is essentially self-adjoint on the $l_0^2(\mathbb{Z})$. This is the case if, for example, the Carleman condition

$$\sum_n \frac{1}{|A_n|} = \infty, \quad \text{where } |A_n| = \max\{a_{-n-2}, a_n\}.$$

is satisfied. For a deeper discussion of the self-adjointness question of Jacobi matrices we refer the reader to [2] (see also [16] for recent results).

The spectral properties of Jacobi matrices are extensively studied by the help of different methods, see for example [5–15,17,19] and references therein. A part of this literature is based on the positive commutator method of Putnam–Kato (cf. [20]). It is used, for example, by Dombrowski [6–8], Dombrowski–Pedersen [9–12], Janas–Moszynski [13] and Pedersen [19]. This paper is a contribution in this direction.

The Putnam–Kato method states that if there is a bounded self-adjoint operator A such that the commutator $[H, iA]$ is positive and injective, then H has a purely absolutely continuous spectrum. This theorem allowed the resolution of great spectral problems [21]. However, the boundedness of A is rather annoying in some applications, since it excludes many natural candidates to this role. The same can be said on the required commutator positivity which is global and then very unstable under perturbation (even compact). The conjugate operator theory of Mourre [1,3,18,23] resolves these problems. The conjugate operator A has not to be bounded anymore. Moreover, the commutator has only to be positive locally and up to a compact operator. The prize to pay is some additional regularity of the underlying operator H with respect to A.

The Mourre method has proved its powerful in the spectral analysis and scattering theory of partial differential operators (see [1]), discrete Schrődinger operators and bounded Jacobi matrices (see [4]), etc. Surprisingly, in the context of unbounded Jacobi matrices Mourre's method has never been used yet. This might create a negative impression on this theory in this context. We even meet in the literature statement explaining that Putnam–Kato's method is superior to Mourre's theory in the context of unbounded Jacobi matrices, see the introduction of [19]. Our purpose in this paper is to show that this is not fully true. We indeed apply the conjugate operator theory to study the spectral and propagation properties of unbounded Jacobi matrices. We focus here on models considered in the quoted literature and for which Putnam–Kato's method does not apply, since the unboundedness of A, see [12,13].

More specifically, in this paper we deal with weights of the following form

$$a_n = |n|^\alpha(1+\eta_n), \; 0 < \alpha < 1 \quad \text{and } \eta_n \to 0, \text{ as } n \to \infty. \tag{1.2}$$

By Carleman condition H is essentially self-adjoint on $l_0^2(\mathbb{Z})$. Consider the operator A defined in $\mathcal{H}$ by

$$(A\psi)(n) = i(\alpha_{n-1}\psi_{n-1} - \alpha_n\psi_{n+1}),$$
$$\alpha_n = \frac{n}{a_n} \quad \text{for } |n| \geqslant N_0,$$

The operator A has been used in the literature for the analysis of Jacobi matrices with weights of the form (1.2) but with null diagonal elements, see for instance [12,13]. Since for $0 < \alpha < 1$ the operator A is unbounded, Putnam–Kato's method does not apply. This explains why only the absence of the point spectrum is proven in [12,13] by a kind of Virial theorem. In this work we show that Mourre's theory applies easily in this context and so we complete these results by proving the absence of singular continuous spectrum for such matrices. The stability of the Mourre estimate under compact perturbation allows us to treat a class of diagonal elements.

Finally, notice that for the case where $\alpha = 1$ the operator A becomes bounded and no Mourre estimate can be expected with such A. This does not mean that Mourre's theory does not apply but only says that we have to change the conjugate operator A. This will be done in a work in progress. For the case where $\alpha = 0$ we get the discrete Schrődinger operator already studied by Mourre's method in [4].

1.2. Main results

Our assumption on η_n and b_n are the following:

$$|\eta_{n+1} - \eta_n| \leqslant C|n|^{-\beta}, \quad \text{with } \beta > 1, \tag{1.3}$$

$$|b_{n+1} - b_n| \leqslant C|n|^{-\delta}, \quad \text{with } \alpha + \delta > 1. \tag{1.4}$$

This means that the discrete derivatives of η_n and β_n decays sufficiently enough, and then η_n and β_n are not so oscillating.

Theorem 1.1. *If* (1.2)–(1.4) *hold then the point spectrum $\sigma_p(H)$ of H is finite and the singular continuous spectrum $\sigma_{\text{sc}}(H)$ of H is empty.*

In particular, if $\eta_n = b_n = 0$ then we are in the situation considered by [12], and if only $b_n = 0$ we get the case considered by [13]. In these papers only the absence of the point spectrum is proven. Notice, however, that Theorem 1.6 of [13] applies to our case but their proof is based on the Gilbert–Pearson theory of subordinate solutions. This is a one-dimensional method while our approach is not, and so our results can be extended to multidimensional models. Furthermore, we derive some a priori estimates for the resolvent of H, that are of physical interest, see the next theorem.

Define the following numbers

$$m = \min(\beta - 1, \alpha + \delta - 1);$$
$$\sigma = m + \frac{1 - \alpha}{2}.$$

Consider the multiplication operator N acting in $\mathcal{H}$ by

$$(N\psi)_n = \sqrt{1+n^2}\psi_n,$$

which is clearly a positive operator in $\mathcal{H}$. For a complex number z we denote by $\Re z$ and $\Im z$ the real and the imaginary parts of z. Put $\mathbb{C}_\pm = \{z \in \mathbb{C} / \pm \Im z > 0\}$. We have the following propagation theorem:

Theorem 1.2. *Assume that* (1.2)–(1.4) *hold. Then the holomorphic maps*

$$\mathbb{C}_\pm \ni z \longmapsto N^{-\sigma}(H-z)^{-1}N^{-\sigma} \in B(\mathcal{H})$$

extends to a locally Hőlder continuous maps on $\mathbb{C}_\pm \cup [\mathbb{R} \setminus \sigma_p(H)]$ *of order* $\theta = \min(\frac{m}{1-\alpha}, \frac{1}{2})$. *Moreover for all* $\varphi \in C_0^\infty(\mathbb{R} \setminus \sigma_p(H))$ *we have*

$$\|N^{-\sigma}e^{-iHt}\varphi(H)N^{-\sigma}\| \leqslant C(1+|t|)^{-\theta}.$$

The paper is organized as follows. In Section 2 we give a brief review on what we need from the conjugate operator theory. Section 3 is completely devoted to case where $b_n = 0$. In Section 4 we prove Theorems 1.1 and 1.2.

2. The conjugate operator method

In this section we present a short description of what we need from Mourre's theory. This is based on [1,23].

Let H, A be two self-adjoint operators in a Hilbert space $\mathcal{H}$. Denote by $\sigma(H)$ the spectrum of H, and for $z \in \mathbb{C} \setminus \sigma(H)$ we set $R_z = (H-z)^{-1}$ the resolvent of H. We equip $D(H)$ with the graph topology associated to the norm

$$\|f\|_H = \|f\| + \|Hf\|.$$

Definition 2.1. Let $0 < s < 1$. We say that H is of class $C^1(A)$ if the map

$$t \longmapsto R_z(t) := e^{-iAt}R_z e^{iAt} \in B(\mathcal{H})$$

is strongly of class C^1 on $\mathbb{R}$ for some $z \in \mathbb{C} \setminus \sigma(H)$. In such case if the derivative is Hőlder continuous of order s (respectively strongly C^1) then we say that H is of class $\mathcal{C}^{1+s}(A)$ (respectively $C^2(A)$).

For $z \in \mathbb{C} \setminus \sigma(H)$ define

$$D(z) := \{\psi \in D(A) / R(z)\psi \in D(A)\}.$$

One has the following characterization of the $C^1(A)$-regularity in terms of the commutator $[H, iA]$ (see [1]):

Proposition 2.1. *The operator H is of class* $C^1(A)$ *if and only if the following two conditions hold*:

(i) *the sets* $D(z)$ *and* $D(\overline{z})$ *are dense in* $D(A)$, *for some* $z \in \mathbb{C} \setminus \sigma(H)$,

(ii) *there is a constant* $c > 0$ *such that for all* $\psi \in D(A) \cap D(H)$:

$$|\langle \psi, [H, iA]\psi \rangle| = |2\Re\langle H\psi, iA\psi\rangle| \leqslant c\|\psi\|_H^2.$$

In particular, if H is of class $C^1(A)$ then $D(A) \cap D(H)$ is dense in $D(H)$ endowed with $\|\cdot\|_H$ and $[H, iA]$ has a unique extension to a continuous sesquilinear form on $D(H)$. Then by Riesz Lemma there exists a unique bounded operator, denoted by the same symbol $[H, iA]$, from $D(H)$ to its adjoint $D(H)^*$ associated to this extension. Let E be the spectral measure of H.

Definition 2.2. Assume that H is of class $C^1(A)$. We say that A is conjugate to H on a real open set Δ if there exist a constant $a > 0$ and a compact operator K in $\mathcal{H}$ such that

$$E(\Delta)[H, iA]E(\Delta) \geqslant E(\Delta) + K. \tag{2.1}$$

The inequality (2.1) is called the Mourre estimate.

Theorem 2.1. *Assume that H is of class* $C^1(A)$ *and that A is conjugate to H on a real open set* Δ. *Then*:

(1) *the operator H has at most a finite number of eigenvalues in* Δ (*counted with multiplicities*).
(2) *if H is of class* $\mathcal{C}^{1+s}(A)$ *for some* $s > 0$ *then H has no singular continuous spectrum in* Δ.

The last statement is a particular case of the main theorem of [23] (see [1] for different versions of this theorem). Let us set $\langle A\rangle = (1 + A^2)^{1/2}$. One has

Theorem 2.2. *Assume that H is of class* $\mathcal{C}^{1+s}(A), 0 < s < 1/2$. *Then the holomorphic maps*

$$\mathbb{C}_\pm \ni z \longmapsto \langle A\rangle^{-(s+\frac{1}{2})}(H - z)^{-1}\langle A\rangle^{-(s+\frac{1}{2})} \in B(\mathcal{H})$$

extends to a locally Hőlder continuous maps on $\mathbb{C}_\pm \cup [\Delta \setminus \sigma_p(H)]$ *of order s. Moreover, for every* $\varphi \in C_0^\infty(\Delta \setminus \sigma_p(H))$ *we have*

$$\|\langle A\rangle^{-(s+\frac{1}{2})} e^{-iHt}\varphi(H)\langle A\rangle^{-(s+\frac{1}{2})}\| \leqslant C\langle t\rangle^{-s}.$$

The restriction $0 < s < 1/2$ comes from the fact that we do not require that H has a spectral gap nor that its domain is stable under the action of the group e^{iAt} (see [23]). Notice that this is the situation of the application under consideration in this paper.

We end this review by a criterion which allows us to check regularity requirement. It is actually a particular case of a theorem obtained in [22] and published in an appendix of [4].

Let $s \in (0, 1)$. We say that a bounded operator T in $\mathcal{H}$ is of class $\mathcal{C}^s(A)$ if the map

$$t \longmapsto T(t) = e^{-iAt}Te^{iAt} \in B(\mathcal{H})$$

is Hőlder continuous of order $s \in (0, 1)$.

In particular, if the operator H is of class $C^1(A)$ and that $B = [H, iA]$ is bounded in $\mathcal{H}$. Then H is of class $\mathcal{C}^{1+s}(A)$ if B is of class $\mathcal{C}^s(A)$.

Theorem 2.3. *Let $\Lambda > 1$ be a self-adjoint operator in $\mathcal{H}$ such that $A\Lambda^{-1}$ is a bounded in $\mathcal{H}$. Then a bounded operator T in $\mathcal{H}$ is of class $\mathcal{C}^s(A)$ if there exists a function $\theta \in C_0^\infty(\mathbb{R})$ with $\theta(x) > 0$ if $0 < a < |x| < b < \infty$ such that*:

$$\sup_{r>1} \|r^s \theta(\Lambda/r) T\| < \infty. \tag{2.2}$$

3. Mourre's estimate for the pure off-diagonal case

This section is entirely devoted to the pure off-diagonal case, i.e. the case where $b_n = 0$. More precisely, consider the operator H_0 acting in $\mathcal{H}$ by

$$(H_0\psi)_n = a_{n-1}\psi_{n-1} + a_n\psi_{n+1}.$$

Let N_0 be a positive integer and consider the operator A defined in $\mathcal{H}$ by

$$(A\psi)(n) = i(\alpha_{n-1}\psi_{n-1} - \alpha_n\psi_{n+1}), \tag{3.1}$$

$$\alpha_n = \frac{n}{a_n} \quad \text{for } |n| \geqslant N_0. \tag{3.2}$$

Clearly A is essentially self-adjoint operator on $l_0^2(\mathbb{Z})$.

For notational convenience denote the first discrete derivative of η_n by

$$\gamma_n := \eta_{n+1} - \eta_n.$$

Assumption (1.3) states that

$$n\gamma_n = O(|n|^{1-\beta}), \beta > 1 \quad \text{at infinity}.$$

For the Mourre estimate we need less than that:

Theorem 3.1. *Assume that* (1.2) *holds and that*

$$\lim_{|n|\to\infty} \gamma_n = 0. \tag{3.3}$$

Then the operator H_0 is of class $C^1(A)$ and A is conjugate to H_0 on $\mathbb{R}$, i.e. there exist a constant $a > 0$ and a compact operator K in $\mathcal{H}$ such that

$$[H, iA] \geqslant a + K.$$

Proof. (1) Let $z \in \mathbb{C} \setminus \mathbb{R}$. Start by showing that the sets

$$D(z) := \{\psi \in D(A)/(H_0 - z)^{-1}\psi \in D(A)\},$$
$$D(\bar{z}) := \{\psi \in D(A)/(H_0 - \bar{z})^{-1}\psi \in D(A)\}$$

are dense in $D(A)$. Let $\{e^k\}_{k\in\mathbb{Z}}$ be the canonical orthonormal basis of $\mathcal{H}$, that is, for each integer k the vector e^k is defined by $(e^k)_n = 0$ for all $n \neq k$ and $(e^k)_k = 1$. For every $k \in \mathbb{Z}$ we define the vector

$$\begin{aligned}\psi^k &= (H_0 - z)e^k \\ &= a_{k-1}e^{k-1} - ze^k + a_k e^{k+1}.\end{aligned} \tag{3.4}$$

Clearly $\psi^k \in D(z)$, since ψ^k and $(H_0 - z)^{-1}\psi^k = e^k$ belong to $l_0^2(\mathbb{Z}) \subset D(A)$. Moreover, $\{\psi^k\}_k$ is linearly independent and span of $\{\psi^k\}_k$ is dense in $\mathcal{H}$. Indeed, let $f \in \mathcal{H}$ such that

$$\langle \psi^k, f\rangle = 0 \quad \forall k \in \mathbb{Z}.$$

According to (3.4) one has for all $k \in \mathbb{Z}$:

$$\begin{aligned}0 = \langle \psi^k, f\rangle &= \langle (H_0 - z)e^k, f\rangle \\ &= a_{k-1} f_{k-1} - \bar{z} f_k + a_k f_{k+1} \\ &= ((H_0 - \bar{z}) f)_k.\end{aligned}$$

It follows that $f \in D(H_0)$ and

$$(H_0 - \bar{z}) f = 0.$$

Since H_0 is a self-adjoint operator and $\bar{z} \in \mathbb{C} \setminus \mathbb{R}$, we deduce that $f = 0$ and so span of $\{\psi^k\}_k$ is dense in $\mathcal{H}$. Hence, by Gram–Schmidt procedure, one constructs an orthonormal basis $\{\tilde{e}^k\}$ of $\mathcal{H}$ from $\{\psi^k\}_k$. Each $\tilde{e}^k$ is a finite linear combination of the ψ^k's and so belongs to $D(z)$. It follows that $D(z)$ contains all the elements of an orthonormal basis of $\mathcal{H}$ and is therefore dense in $D(A)$ (since A is closed). The same can be done for $D(\bar{z})$ by substituting z to $\bar{z}$.

(2) To show that H_0 is of class $C^1(A)$ it remains to verify the point (ii) of the Proposition 2.1. Direct computation shows that for $\psi \in l_0^2(\mathbb{Z})$ we have

$$\langle \psi, [H_0, iA]\psi\rangle = \sum_n \overline{\psi}_n (c_{n-2}\psi_{n-2} + 2d_n\psi_n + c_n\psi_{n+2}), \tag{3.5}$$

where for $|n| \geqslant N_0$ we have

$$\begin{aligned}d_n &= 1; \\ c_n &= a_{n+1}\alpha_n - a_n\alpha_{n+1} \\ &= n\frac{a_{n+1}}{a_n} - (n+1)\frac{a_n}{a_{n+1}}.\end{aligned}$$

We obviously have

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \left(1 + \frac{\alpha}{n} + O(1/n^2)\right)(1 + \gamma_n - \gamma_n O(\eta_n)) \\ &= 1 + \frac{\alpha}{n} + \gamma_n + \gamma_n O(\eta_n), \\ \frac{a_n}{a_{n+1}} &= 1 - \frac{\alpha}{n+1} - \gamma_n + \gamma_n O(\eta_n).\end{aligned}$$

It follows that

$$c_n = n\frac{a_{n+1}}{a_n} - (n+1)\frac{a_n}{a_{n+1}} = -1 + 2\alpha + (2n+1)\gamma_n + \gamma_n O(\eta_n)$$
$$= -1 + 2\alpha + \varepsilon_n.$$

Since $\eta_n \to 0$ and $n\gamma_n \to 0$ as n tends to infinity (by our hypothesis), $\varepsilon_n \to 0$ too. In particular, the quadratic form $[H_0, iA]$ can be extended to a continuous quadratic form in $\mathcal{H}$. Combining this with the point (1) of the present proof we conclude that H_0 is of class $C^1(A)$, and the associated operator to $[H_0, iA]$, which will denote by the same symbol, is a bounded operator in $\mathcal{H}$.

(3) From the last computation we further deduce the identity

$$[H_0, iA] = 2 + (2\alpha - 1)T + K, \tag{3.6}$$

where the operators T and K acts in $\mathcal{H}$ as follows

$$(T\psi)_n = \psi_{n-2} + \psi_{n+2}, \tag{3.7}$$

$$(K\psi)_n = \varepsilon_{n-2}\psi_{n-2} + \varepsilon_n\psi_{n+2}. \tag{3.8}$$

On one hand, the operator T is bounded and

$$\|T\| \leqslant 2.$$

On other hand, since $\varepsilon_n \to 0$ as $|n| \to \infty$, the operator K is compact. Then

$$[H_0, iA] \geqslant 2(1 - |2\alpha - 1|) + K.$$

Since $\alpha \in (0, 1)$, the number

$$a = 2(1 - |2\alpha - 1|) > 0.$$

In other words,

$$[H_0, iA] \geqslant a + K, \tag{3.9}$$

which finishes the proof of the theorem. □

Remark. According to the first point of Theorem 2.1 the operator H_0 has at most a finite number of eigenvalues. Emphasis the fact that this has been obtained without specifying the matrix elements α_n of A for $|n| \leqslant N_0$. This can be done in such a way that (cf. [13]) the commutator $[H_0, iA] > 0$ which implies that H_0 has no eigenvalues. Indeed, since H_0 is of class $C^1(A)$ the Virial lemma holds (cf. [1]):

$$\langle f, [H_0, iA]f\rangle = 0 \quad \text{for every eigenvector } f \text{ of } H.$$

This is also the main argument of [12,13], but the authors of these papers made considerable efforts to establish it.

Proposition 3.1. *The operator K is of class* $\mathcal{C}^s(A)$, $s = \frac{\beta-1}{1-\alpha} > 0$.

The proof of this proposition is based on Theorem 2.3 and the following obvious lemma

Lemma 3.1. *Let Λ be the positive operator defined by*

$$(\Lambda\psi)_n = (1+|\alpha_n|)\psi_n.$$

The operator $\Lambda^{-1}A$ is bounded in $\mathcal{H}$.

Proof of Proposition 3.1. According to Theorem 2.3, K is of class $\mathcal{C}^s(A)$, for some $s > 0$, if

$$\sup_{r>0} r^s\|\theta(\Lambda/r)K\| < \infty.$$

for some function $\theta \in C_0^\infty(\mathbb{R})$ with $\theta(x) > 0$ if $0 < a < |x| < b < \infty$. But

$$\begin{aligned} r^s\|\theta(\Lambda/r)K\| &\leqslant r^s\|\theta(\Lambda/r)\Lambda^s\Lambda^{-s}K\| \\ &\leqslant r^s\|\theta(\Lambda/r)\Lambda^{-s}\|\cdot\|\Lambda^sK\| \\ &\leqslant Cr^sr^{-s}\|\Lambda^sK\| = C\|\Lambda^sK\|. \end{aligned} \tag{3.10}$$

On the other hand, according to the proof of Theorem 3.1 all the matrix elements ε_n of K behaves for n large enough as follows

$$\varepsilon_n = O(n\gamma_n).$$

Hypothesis (1.3) implies that $\varepsilon_n = O(|n|^{-(\beta-1)})$ at infinity. Hence for some N large enough one has

$$\|\Lambda^sK\| \leqslant C(1+\sup_{|n|>N}|n|^{s(1-\alpha)}|n|^{-(\beta-1)}) < \infty,$$

provided that $s = \frac{\beta-1}{1-\alpha} > 0$. □

Corollary 3.1. *The operator H_0 is of class $\mathcal{C}^{1+s}(A)$, $s = \frac{\beta-1}{1-\alpha} > 0$.*

Proof. According to the proof of Theorem 3.1 we have

$$B_0 = 2 + (2\alpha-1)T + K.$$

So B is of class $\mathcal{C}^s(A)$ if each component T and K are. On one hand, according to the preceding lemma K is of class $\mathcal{C}^s(A)$. It is enough then to show that the bounded operator T is of class $\mathcal{C}^s(A)$. We will prove in fact that T is of class $C^1(A)$. Indeed, it is not difficult to get

$$\begin{aligned} ([T,iA]\psi)_n = &(\alpha_n-\alpha_{n+2})\psi_{n+3} + (\alpha_{n+1}-\alpha_{n-1})\psi_{n+1} \\ &+(\alpha_n-\alpha_{n-2})\psi_{n-1} + (\alpha_{n-3}-\alpha_{n-1})\psi_{n-3}. \end{aligned}$$

But obviously

$$\alpha_n - \alpha_{n+2} = O(|n|^{-\alpha}) \quad \text{as } |n| \to +\infty.$$

In particular, $[T,iA]$ can be extended to a bounded operator in $\mathcal{H}$ and so T is of class $C^1(A)$. □

From Theorem 2.1 one has

Corollary 3.2. *The spectrum of H_0 is purely absolutely continuous on $\mathbb{R}$ up at most a finite number of eigenvalues.*

4. Proof of Theorem 1.1

Theorem 1.1 deals with the operator H which we rewrite as follows

$$H = H_0 + b,$$

where b is the multiplication operator by the sequence $(b_n)_n$. Consider the operator A defined by (3.1) and (3.2).

Proposition 4.1. *Suppose that*

$$\lim_{n\to\infty} |n|^{1-\alpha}(b_{n+1} - b_n) = 0. \tag{4.1}$$

Then there exists a constant $C > 0$ such that for any $\varphi, \psi \in l_0^2(\mathbb{Z})$ we have

$$\|\langle\varphi, [b, iA]\psi\rangle| \leqslant C\|\varphi\| \cdot \|\psi\|.$$

Moreover, the associated bounded operator $[b, iA]$ is compact in $\mathcal{H}$.

Proof. It is easy to see that $\tilde{b} = [b, iA]$ is given, in form sense on $l_0^2(\mathbb{Z})$, by

$$(\tilde{b}\psi)_n = \alpha_{n-1}(b_n - b_{n-1})\psi_{n-1} + \alpha_n(b_{n+1} - b_n)\psi_{n+1}.$$

But according to our hypothesis

$$\tilde{b}_n = \frac{n}{a_n}(b_{n+1} - b_n) \to 0 \quad \text{as } |n| \to \infty.$$

So the assertion of the proposition follows immediately. □

Corollary 4.1. *Assume that* (3.3) *and* (4.1) *hold. Then the operator H is of class $C^1(A)$ and A is conjugate to H on $\mathbb{R}$.*

Proof. Combining the preceding proposition with the results of the last section, we get that there exists a constant $C > 0$ such that for any $\varphi, \psi \in l_0^2(\mathbb{Z})$ we have

$$\begin{aligned} |\langle\varphi, [H, iA]\psi\rangle| &= |\langle\varphi, [H_0, iA]\psi\rangle| + |\langle\varphi, [b, iA]\psi\rangle| \\ &\leqslant C\|\varphi\| \cdot \|\psi\|. \end{aligned} \tag{4.2}$$

Moreover, one can prove that the sets $D(z)$ and $D(\overline{z})$ are dense in $D(A)$ as we did for H_0. Hence H is of class $C^1(A)$ and $[H, iA]$ is a bounded operator in $\mathcal{H}$. Moreover one has

$$\begin{aligned} [H, iA] &= [H_0, iA] + [b, iA] \\ &\geqslant a + K_1, \end{aligned}$$

where $a = 2(1 - |2\alpha - 1|) > 0$ and $K_1 = K + [b, iA]$ which is a compact operator in $\mathcal{H}$. This shows that the Mourre estimate holds on $\mathbb{R}$. □

Remark. In particular, H has at most a finite number of eigenvalues.

Proposition 4.2. *Under the hypothesis of Theorem* 1.1 *the operator H is of class* $\mathcal{C}^{1+s}(A)$, $s = m/(1-\alpha) > 0$.

Proof. We saw that $[H, iA] = [H_0, iA] + [b, iA]$ is a bounded operator in $\mathcal{H}$. Moreover, On one hand we have $[H_0, iA]$ is a of class $\mathcal{C}^{(\beta-1)/(1-\alpha)}(A)$. On the other hand, $[b, iA]$ is of class $\mathcal{C}^{(\alpha+\delta-1)/(1-\alpha)}(A)$. This can be shown by the same argument as for Proposition 3.1, since all matrix elements of $[b, iA]$ decay as $|n|^{1-\alpha-\delta}$. So, $[H, iA]$ is of class $\mathcal{C}^s(A)$, $s = \min(\frac{\beta-1}{1-\alpha}, \frac{\alpha+\delta-1}{1-\alpha})$. This means that H is $\mathcal{C}^{1+s}(A)$. We conclude the proof by noting that according to the definition of the number m in the introduction we have $s = m/(1-\alpha)$. $\square$

Proof of Theorem 1.1–1.2. Theorem 1.1 follows from Corollary 4.1, Proposition 4.2 combined with Theorem 2.1.

Again from Corollary 4.1, Proposition 4.2 combined with Theorem 2.2 we get that the holomorphic maps

$$\mathbb{C}_\pm \ni z \longmapsto \langle A\rangle^{-(s+\frac{1}{2})}(H-z)^{-1}\langle A\rangle^{-(s+\frac{1}{2})} \in B(\mathcal{H})$$

extends to a locally Hőlder continuous maps on $\mathbb{C}_\pm \cup [\mathbb{R}\setminus\sigma_p(H)]$ of order $\theta = \min(\frac{m}{1-\alpha}, \frac{1}{2})$. But

$$\langle A\rangle^{(s+\frac{1}{2})} N^{-(s+\frac{1}{2})(1-\alpha)} \text{ is a bounded operator in } \mathcal{H}.$$

And by the definition of the number σ in the introduction one has

$$\left(s+\frac{1}{2}\right)(1-\alpha) = m + \frac{1-\alpha}{2} = \sigma.$$

Hence the maps which to $z \in \mathbb{C}_\pm$ associate

$$N^{-\sigma}(H-z)^{-1}N^{-\sigma} = N^{-\sigma}\langle A\rangle^{(s+\frac{1}{2})}[\langle A\rangle^{-(s+\frac{1}{2})}(H-z)^{-1}\langle A\rangle^{-(s+\frac{1}{2})}] \cdot \langle A\rangle^{(s+\frac{1}{2})}N^{-\sigma}$$

clearly extend to a Hőlder continuous maps on $\mathbb{C}_\pm \cup [\mathbb{R}\setminus\sigma_p(H)]$ of order θ.

The second assertion of the theorem follows by a similar argument from

$$N^{-\sigma}e^{-iHt}\varphi(H)N^{-\sigma} = N^{-\sigma}\langle A\rangle^{(s+\frac{1}{2})}[\langle A\rangle^{-(s+\frac{1}{2})}e^{-iHt}\varphi(H)\langle A\rangle^{-(s+\frac{1}{2})}] \cdot \langle A\rangle^{(s+\frac{1}{2})}N^{-\sigma}$$

The proof is finished. $\square$

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